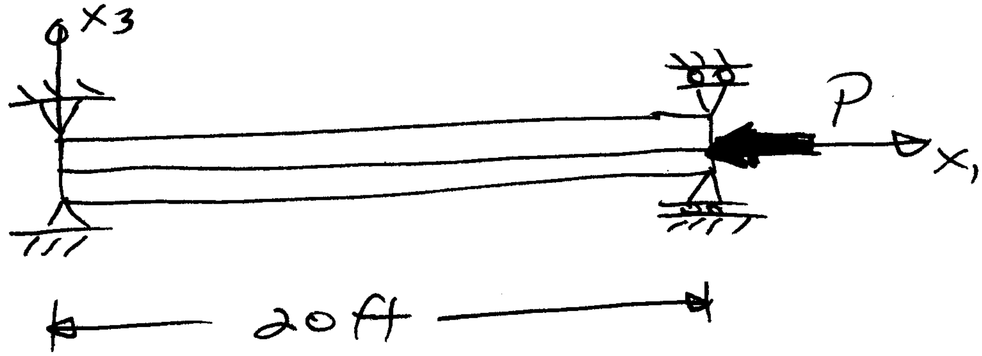
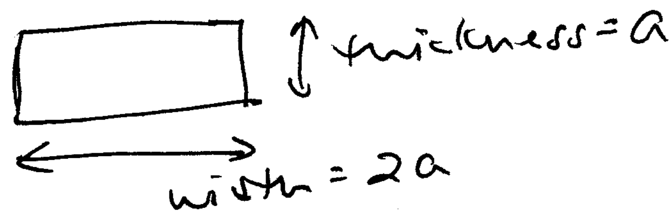
PAL
4/10/08

Unified Engineering Problem Set
Week #11 Spring, 2008
SOLUTIONS

M 11.1



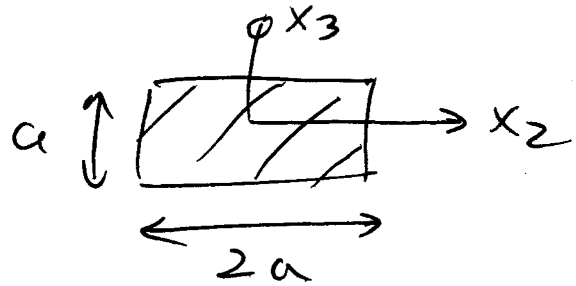
Cross-Section:



(a) Model this as a simply-supported column. For such a configuration:

$$P_{cr} = \frac{\pi^2 E I}{L^2}$$

Buckling will occur perpendicular to the axis about which I is a minimum. For a rectangular cross-section, that will be along the direction of the smallest cross-sectional dimension -- the thickness (= a) in this case. So align that with the x_3-axis and:



$$I = \frac{bh^3}{12} = \frac{(2a)(a^3)}{12} = \frac{a^4}{6}$$

The value of modulus E, for titanium in this case is 18.5×10^6 lbs/in^2.

The length of the column is 20 feet and converting to inches gives: L = 240 in

Using these values in the expression for P_{cr} gives:

$$P_{cr} = \frac{\pi^2 \left(18.5 \times 10^6 \frac{\text{lbs}}{\text{in}^2}\right)\left(a^4/6\right)}{(240 \text{ in})^2}$$

This gives:

$$\boxed{P_{cr} = 528.3\, a^4}$$ a in [in]
P in [lb]

(b) To determine the squashing load the material compressive ultimate is needed. For titanium:

$$\sigma_{cu} = 150 \text{ ksi}$$

This is related to column load via the column area:

$$\frac{P_{squash}}{A} = \sigma_{cu}$$

Here, the area is: $A = (a)(2a) = 2a^2$

Progressing:

$$P_{squash} = \sigma_{cu}\, 2a^2$$

$$\Rightarrow \boxed{P_{sq} = 300{,}000\, a^2}$$ a in [in]
P in [lb]

One can also determine the start of a "transition" zone via:

$$\frac{P_{transition}}{A} = \sigma_{cy}$$

For the titanium, $\sigma_{cy} = 98$ ksi

$$\Rightarrow \boxed{\underset{(yield)}{P_{trans}} = 196{,}000\, a^2}$$

a in [in]
P in [lb]

(c) The key to drawing the design chart is to determine the points (P and a) where the mode of failure goes from "buckling" to "transition" to "crushing/squashing". Do this by equating the buckling case with the latter two for each case, solving for a, and substituting the result back into the governing equation to get the associated value of P. Then plot each curve.

Summarizing what we have:

(A) Buckling: $P_{cr} = 528.3\, a^4$

(B) Transition: $P_{tran} = 196{,}000\, a^2$
(yielding)

(C) Squashing: $P_{sq} = 300{,}000\, a^2$

All a in [in], P in [lb]

→ going from (A) to (B):

$$528.3\, a^4 = 196{,}000\, a^2$$

$$\Rightarrow a^2 = 371.0$$

$$\Rightarrow a = 19.3 \text{ in}$$

which gives $P = 7.33 \times 10^7$ lbs

→ going from (A) to (C):

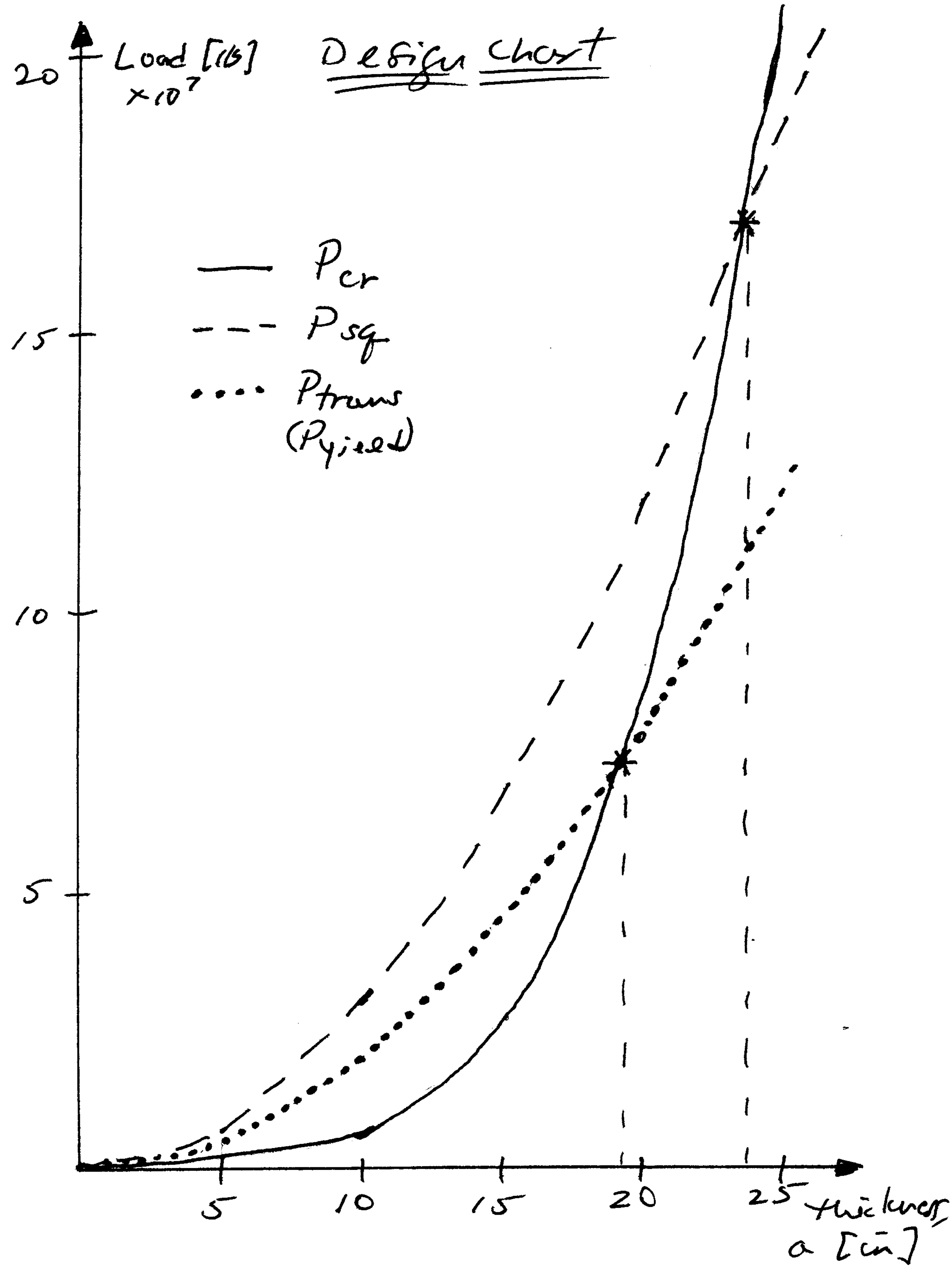
$$528.3\, a^4 = 300{,}000\, a^2$$

$$\Rightarrow a^2 = 567.9$$

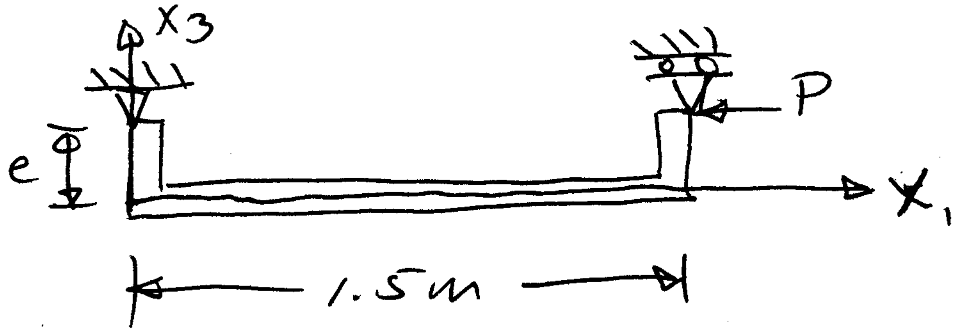
$$\Rightarrow a = 23.8 \text{ in}$$

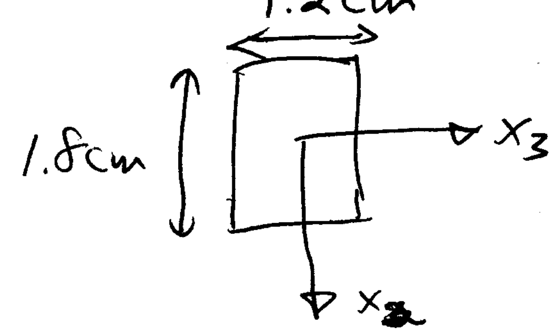
which gives $P = 1.70 \times 10^8$ lbs

Now draw the plots of each curve and label these key points


Load [lb] × 10^7
Design chart
20
15
10
5
Pcr
Psq
Ptrans
(Pyield)
5
10
15
20
25
thickness, a [in]

M 11.2





(a) The maximum load is the limit placed by the buckling load <u>or</u> by the transition or squashing load. This can occur for the case of no eccentricity ($e=0$). With <u>eccentric</u> loading, that maximum value does not change. However the eccentricity may result in it not being reached.

For a simply-supported configuration as is the case in this problem, the critical buckling load is:

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

with I determined for its minimum value based on the cross-section. For a rectangular case, that is:

$$I = \frac{bh^3}{12}$$

where in this case: $b = 1.8\,cm$; $h = 1.2\,cm$

$$\Rightarrow I = \frac{(1.8\,cm)(1.2\,cm)^3}{12} = \frac{(1.8\times10^{-2}\,m)(1.2\times10^{-2}\,m)^3}{12}$$

$$= 2.59\times10^{-9}\,m^4$$

The material in this case is aluminum with a modulus, E, of $70\,GPa = 70\times10^9\,\frac{N}{m^2}$

with $L = 1.5\,m$, use these values in the expression for P_{cr}:

$$P_{cr} = \frac{\pi^2\left(70\times10^9\,\frac{N}{m^2}\right)\left(2.59\times10^{-9}\,m^4\right)}{(1.5\,m)^2}$$

$$\Rightarrow P_{cr} = 796 \text{ N}$$

Check σ_{cr} to see if it is below or above σ_{cy} and σ_{cu}:

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{796 \text{ N}}{(1.8\times10^{-2} \text{ m})(1.2\times10^{-2} \text{ m})} = 3.69\times10^{6} \frac{\text{N}}{\text{m}^2}$$

$$= 3.69 \text{ MPa}$$

and this is well below the yield and ultimate stress

Thus, P_{cr} is the maximum load the column can carry

$$\boxed{\text{Max load} = P_{cr} = 796 \text{ N}}$$

(b) For the case of a simply-supported configuration loaded eccentrically, the governing equation is:

$$u_3 = e\left[\frac{\left(1-\cos\sqrt{\frac{P}{EI}}\,L\right)}{\sin\sqrt{\frac{P}{EI}}\,L}\sin\sqrt{\frac{P}{EI}}\,x_1 + \cos\sqrt{\frac{P}{EI}}\,x_1 - 1\right]$$

Now use the pertinent values of P_{cr}, E, I, and L, and to determine the deflection at the column center, set $x_1 = 0.75m$.

Normalize that deflection by the column length and the applied load by the critical buckling load. To do this....

multiply P by $\frac{P_{cr}}{P_{cr}} = \frac{\pi^2 EI}{P_{cr}L^2}$

$$\Rightarrow \sqrt{\frac{P}{EI}} = \sqrt{\frac{P}{EI}\cdot\frac{\pi^2 EI}{P_{cr}L^2}} = \sqrt{\frac{P}{P_{cr}}\,\frac{\pi^2}{L^2}}$$

$$\text{so: } \sqrt{\frac{P}{EI}} = \frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}$$

Put this back into the previous equation for u_3 to get:

$$u_3 = e\left[\frac{1-\cos\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}L\right)}{\sin\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}L\right)}\sin\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}x_1\right)+\cos\left(\frac{\pi}{L}\sqrt{\frac{P}{P_{cr}}}x_1\right)-1\right]$$

Continuing on and dividing through by L:

$$\frac{u_3}{L} = \frac{e}{L}\left[\frac{1-\cos\left(\pi\sqrt{\frac{P}{P_{cr}}}\right)}{\sin\left(\pi\sqrt{\frac{P}{P_{cr}}}\right)}\sin\left(\pi\sqrt{\frac{P}{P_{cr}}}\frac{x_1}{L}\right)+\cos\left(\pi\sqrt{\frac{P}{P_{cr}}}\frac{x_1}{L}\right)-1\right]$$

At the center: $\frac{x_1}{L} = 0.5$, so at the center:

$$\boxed{\frac{u_3}{L} = \frac{e}{L}\left[\frac{1-\cos\left(\pi\sqrt{\frac{P}{P_{cr}}}\right)}{\sin\left(\pi\sqrt{\frac{P}{P_{cr}}}\right)}\sin\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)+\cos\left(\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right)-1\right]}$$

This expression does <u>not</u> change with value of eccentricity, only the magnitude of the eccentricity, e.

(c) Use this expression to make plots for the five cases of $\frac{e}{L} = 0, 0.01, 0.02, 0.05, 0.1$.

This give an asymptote from the behavior for $e/L = 0$ of no deflection for no load and deflection to infinity at P_{cr}.

Normalized Load vs. Normalized Center Deflection